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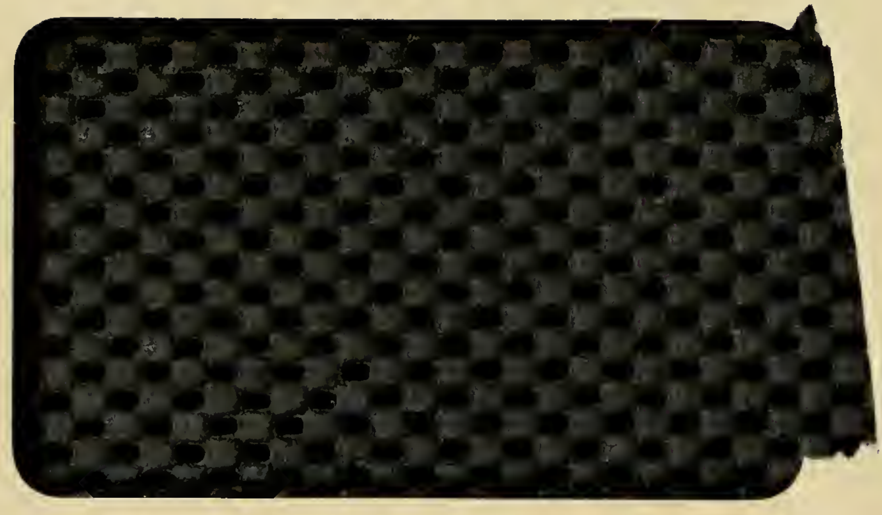
EQUATIONS OF FLOW IN A RAREFIED ATMOSPHERE

Harold Grad

June 17, 1959

Institute of Mathematical Sciences

NEW YORK UNIVERSITY
NEW YORK, NEW YORK



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EQUATIONS OF FLOW IN A RAREFIED ATMOSPHERE*

Harold Grad

ABSTRACT

The problem of the aerodynamic or flow properties of the upper atmosphere is characterized by the presence of a large number of simultaneous and interacting phenomena and a vast range of values taken by the relevant physical parameters with an ensuing bewildering array of different flow regimes. As a working rule, one can hope to develop a non-trivial mathematical theory if at most one physical parameter (e.g. the Mach number) is allowed to range over all its values, and all other parameters (e.g. the mean-free-path) are taken to be either very small or very large. In the first part of this paper, we present a survey of some of the more interesting limiting cases, and we mention only briefly the multitude of transition regimes. In the second part we consider, as an illustrative problem, the "free flow" around an object much smaller than the mean-free-path. Surprisingly, whether or not this problem should be handled by the traditional free molecular flow techniques, or by continuum techniques, or as a problem in transition flow turns out to depend on what question has been asked. In particular, one can find in this single

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* This is the text of a paper which was presented at a symposium on the Aerodynamics of the Upper Atmosphere at the Rand Corporation on June 8 - 10, 1959.

problem all the features of the various flow regimes for arbitrary Knudsen number as well as Mach number. The asymptotic flow at infinity can be described by an appropriate solution of the macroscopic non-dissipative Euler equations including a wake behind the body and a shock wave in the supersonic case. The fine structure of the wake is similar to the classical Oseen solution. Exactly the same asymptotic behavior is obtained for the flow around a body of arbitrary size (not in free flow). Thus the asymptotic flow depends on Mach number only and not on Knudsen number.

1. SURVEY OF THE FLOW REGIMES

In the problem of the flow of an ordinary (unionized) gas around an object there are essentially two relevant parameters, the mean-free-path (Knudsen number) and the speed (Mach number). The two extremes of small and large mean-free-path have been analyzed in some detail. In the former case we choose one of the conventional formulations of continuum gas dynamics, the simplest being the non-dissipative Euler equations. For the latter we solve the simple kinetic theory problem of non-interacting molecules which follow straight line paths. The only complication is the selection of an appropriate boundary condition. Varying the flow speed presents no difficulty in the free-flow extreme. It presents considerable difficulty in the continuum formulation, but there exists an extensive literature which covers this subject in detail. The transition regime with regard to mean-free-path is much less understood than the transition with regard to Mach number and, for the flow around a finite object, has hardly been treated except in the low speed limit (see Refs. (1),(2); also (3),(4) and additional Refs. quoted there).

The introduction of ionization opens a veritable Pandora's box of complex phenomena. To start, consider the mean-free-path which is no longer unique and does not, by itself, give a criterion for either free flow or simple continuum flow. There is a mean-free-path for neutral-neutral encounters, for encounters between charged particles, and for encounters between a neutral and a charged particle. For simplicity let us ignore inelastic effects such as ionization, recombination, charge exchange, and related surface effects. We speak of free paths rather than collision times since the former are about the same size for electrons and for ions. We also choose to ignore as minor the difference between neutral-neutral and neutral-charged cross-sections. This leaves two distinct mean-free-paths, the neutral and the Coulomb. At the temperatures existing in the atmosphere, the Coulomb cross-section is much larger than the neutral cross-section. This is counterbalanced by the fact that the neutral density is usually much larger than the charged particle density. The Coulomb mean-free-path is always large (at least 300 meters), but the neutral mean-free-path can range from microscopic values at low altitudes to values large compared to the Coulomb mean-free-path.

In contrast to the situation for neutral gas, we shall see that neither does a small neutral mean-free-path (concomitant with a small mean-free-path for charged-neutral encounters) validate a simple continuum theory, nor do large values of both the neutral and Coulomb mean-free-paths imply that free flow prevails. The explanation lies in the long-range nature of Coulomb forces. In one aspect, the relevant dimension is the Debye shielding length, d. For our purposes, the significance of this length is that removal of all the charged particles of one sign within a cube of side d will create an appreciable electric field (i.e. a potential comparable to kT). The charge removed is nd^3e and the potential at a distance d is $nd^3e^2/\varkappa_o d$ which, set equal to kT, yields the result

$$d^2 = \frac{\varkappa_o kT}{ne^2} . \tag{1.1}$$

If the object is much smaller than the Debye length as well as the mean-free-path, then the perturbation of the flow induced by the object includes a negligible electric field perturbation, and the charged particle orbits can be computed as in free flow. On the other hand, if the Debye length is small compared to macroscopic dimensions, then we can conclude that the ion and electron densities are almost equal everywhere in the flow, and we can use greatly simplified single-fluid macroscopic equations. Of course, this quasi-neutrality of the medium does not imply absence of electric fields; on the contrary, any but enormous fields can arise by only very slight deviations from neutrality.

In the transition regime (finite Debye length) we have a so-called self-consistent field problem. The particle orbits are to be computed in an assumed electric field which must them be made consistent with the solution of Poisson's equation in which the charge source is determined by the particle motions (e.g. see Ref. (5)). For a typical density of 10^5 electrons per cm^3 in the ionosphere, the Debye length is on the order of several millimeters.

An analogous effect arises from the magnetic field. Taking the maximum current density to be that resulting from all particles of one sign moving at thermal speed, $J = ne\sqrt{kT/m}$, we compute a minimum transition distance, δ_1, over which an appreciable magnetic field can be produced by setting the resulting magnetic energy $B^2/2\mu$ equal to the gas energy p where $J \sim B/\mu\delta_1$. We obtain (using $\mu\varkappa = 1/c^2$)

$$\delta_1{}^2 = \frac{mc^2\varkappa_0}{ne^2} = \frac{mc^2}{kT}\,d^2 \ . \tag{1.2}$$

If δ_1 is large compared to the size of the object, then the magnetic field perturbation can be ignored and we have free flow relative to the magnetic field. If δ_1 is small compared to macroscopic dimensions, then we conclude that the mean flow speeds of the ion and electron gases are approximately equal. Even very large currents are, in this case, easily produced by small relative velocities. In the transition regime, one must compute the flow of the two fluids in an assumed magnetic field which is to be consistent with the solution of Maxwell's equations containing the related current source. From the formula (1.2), it is clear that the electric field criterion for free flow is the more stringent, whereas the magnetic criterion for continuum flow is the more stringent. A typical value of δ_1 in the ionosphere is about 10 meters.

In the presence of an external magnetic field, B_o, one can compute an alternative distance, δ_2, as the minimum distance over which the induced magnetic field can be comparable to the external field. We obtain

$$(1.3) \qquad \delta_2 = \delta_1 / \sqrt{\beta} = \lambda/\beta$$

where

$$(1.4) \qquad \lambda = \frac{m\sqrt{kT/m}}{eB_o}$$

is a mean Larmor radius and

$$(1.5) \qquad \beta = \frac{nkT}{B_o^2/2\mu}$$

is the ratio of gas to magnetic energy. In the presence of an external field, δ_1 can also be compared to the Larmor radius, $\delta_1 = \lambda / \sqrt{\beta}$. In the special case of equipartition of magnetic and kinetic energy, $\beta = 1$, we have $\delta_1 = \delta_2 = \lambda$.

In general, we distinguish δ_1, which is the minimum distance required for induced currents to have an effect on the gas, from δ_2, which is the distance over which there can be an effect on the ambient magnetic field. In the ionosphere, the energy density of the earth's field is about 10^6 times the kinetic energy of the ionized component, and it becomes comparable to the total gas energy at about 135 km.

There is another case similar to the ones described by the lengths δ_1 and δ_2. Suppose β is small and δ_1 is large (as in the ionosphere). The induced currents are not large enough to perturb the ambient magnetic field, nor can they create a large enough magnetic pressure to be comparable to the gas pressure, p. However, the force on the ionized component is measured by the product of the small induced current and the relatively large ambient field, B_o. The criterion that this force be appreciable when compared to p yields a length $\delta_3 = \lambda$. The mean electron Larmor radius can have a value from one to several centimeters in the ionosphere.

The relative sizes of the electric and magnetic Lorentz forces, qE and J x B_o, depend on whether the electric field is governed by the Debye length (as in the boundary layer surrounding the body) or by the size of the object (in the main flow). In the boundary layer, the electric force dominates by the ratio λ/d , whereas in the main flow, the magnetic force dominates by the ratio of the cube of a macroscopic dimension to the product λd^2.

The next length we consider is the Larmor radius in its primitive interpretation in terms of particle orbits. In the free flow regime, this offers complications very similar to those existing in conventional free flow problems in which there occur multiple reflections with the boundary. One obtains an integral equation for the incident mass flow on the surface of the object.

In the continuum regime, the comparative sizes of Larmor radius and mean-free-path (both small) enter into the size of the Hall effect term in Ohm's law. Ohm's law can be written in the form

$$E + u \times B = R_{\parallel} J_{\parallel} + \eta J \times B + R_{\perp} J_{\perp} \qquad (1.6)$$

where

$$(1.7) \qquad J_{\parallel} = \frac{1}{B^2}(J\cdot B)B \ , \quad J_{\perp} = \frac{1}{B^2}(B \times J) \times B.$$

$$J = J_{\parallel} + J_{\perp}$$

The resistivity is essentially proportional to the collision frequency or reciprocal mean-free-path, while the Hall coefficient, ηB, is proportional to the Larmor frequency or reciprocal Larmor radius. For a fully ionized two-component gas (one species of ion) the Hall effect arises as a consequence of choosing the "total" fluid velocity, u, as the mass flow velocity; the effect would not be present (at least not in the expression of Ohm's law) if it were taken as the arithmetic mean of the ion and electron gas speeds.

The parameter $R_{\parallel}$ is independent of B while $R_{\perp}$ and η depend only slightly on B; (they may vary on the order of a factor two when B varies from zero to infinity). For a fully ionized two-component gas $R_{\perp}$ is found to be equal to $R_{\parallel}$ when thermal diffusion (i.e. thermoelectricity) is ignored. Inclusion of thermal diffusion or a mixture of ionic species yields unequal values for $R_{\parallel}$ and $R_{\perp}$.

There has been a certain oversimplification in the statement that continuum flow (by which we mean a conventional one-fluid set of equations combined with Maxwell's equations and Ohm's law) results when the mean-free-paths together with d and δ can be considered to be small. One usually considers the conditions that mean-free-path, Larmor radius, and Debye length be macroscopically small to be roughly equivalent to the requirements that collision frequency, Larmor frequency, and plasma frequency respectively be large compared to relevant macroscopic frequencies. Implied in this equivalence is the existence of a macroscopic speed which relates a length to the appropriate frequency. This is not so for the case of plasma oscillations which have the peculiar property that the frequency rather than the phase speed is approximately independent of wavelength. In particular, it is possible to consider plasma oscillations as being in a sense macroscopic, i.e. with a macroscopic wave length much larger than the Debye length, even though the frequency is usually too high to be considered within the province of fluid dynamics. Plasma oscillations are incorporated into a macroscopic theory in which the Debye length has been removed by taking an inertial form of Ohm's law in which the time derivative $\partial J/\partial t$ is kept.

Summarizing, we would recommend that a macroscopic single-fluid theory be used to describe the flow near an object when the mean-free-paths and lengths d and δ are small compared to the size of the object. There is a range (up to 100 km) where the neutral mean-free-path is small compared to a body size of, say, one meter. The charged particle flow would be dominated by the charged-neutral mean-free-path which is about the same size as the neutral mean-free-path. The Debye length and the electron Larmor radius (δ_3) are small, but δ_1 is fairly large and δ_2 is very large. One can therefore use a simplified version of the continuum equations in which the earth's magnetic field is considered to be given and is unaffected by the object, but the magnetic force on the charged component cannot be ignored. However, since the neutral gas pressure is much larger than the ionized gas pressure, this magnetic force can be neglected in the total flow equations and enters only as an additional Hall term in Ohm's law.

Above some 150 km, all mean-free-paths are large. Since the Debye length is still small, one must use a self-consistent electric field analysis for the flow of the charged components. The magnetic forces can be important in perturbing the flow, but they arise in the form $J \times B_o$ where B_o is the unperturbed magnetic field. From our assumptions we would conclude that, in this region, the neutral particle flow and the charged particle flow could be solved independently. However, this is true only insofar as they are not related by surface effects and by inelastic collisions.

2. FREE FLOW REEXAMINED

Consider the free flow of a neutral gas about a sphere. The conventional solution can be described as follows. Given a point in space, x, and a molecular velocity, ξ. The number density of particles (i.e. the distribution function) is the same as at infinity if a line traced backwards with this velocity does not intersect the sphere; the number density is equal to the number of particles leaving the sphere with this velocity if the path traced backwards does intersect the sphere. Thus the distribution function can be considered to be the sum of two distribution functions. First there is the unperturbed distribution function at infinity. Then there is a function which is zero except on the cone subtended by the sphere. This is the difference between the particles emitted by the sphere and those shielded by the sphere from the initial flow.

This description of the flow implies that the mean-free-path has been set equal to infinity. To be more precise, let us consider a case in which the ratio a/L of the radius of the sphere to the mean-free-path is very small but not zero. The elementary free flow solution described above is approximately valid for a distance r from the sphere which is small compared to L; this may be many sphere radii. At a distance comparable to L, the emerging stream will begin to encounter the incident flow and both will be altered. The emerging stream will be deflected back and eventually join the main stream while shielding the body from the incident stream which is somewhat deflected around the body. Clearly, the elementary free flow solution is not uniformly valid for large distances from the sphere. However, since the free flow perturbation decays as $(a/r)^2$ and is small on the order of $(a/L)^2$ by the time collisions become effective, one can say that the free flow solution is accurate in the entire region where there is a significant perturbation of the original flow. Nevertheless, there is some interest in determining the ultimate behavior of the flow, far from the object. This requires a full treatment of the collision problem and the Boltzmann equation. One can expect free flow to prevail near the sphere, continuum flow at a large distance, and a complex transition in between.

This question is related to the more complex problem of the transition of the flow near the body from free flow to continuum flow as (a/L) increases. In particular, for this problem one would like to see how a detached shock (for supersonic flow), and a viscous boundary layer, and possibly a wake behind the body develop. A striking result that we find is that the "detached" shock and wake exist even in the free flow limit of very small (a/L) provided that we look for them far from the body. This is in contrast to the near flow which does not distinguish at all between subsonic and supersonic speeds. Succinctly, the near flow sees no important qualitative dependence on Mach number, while the far flow (we shall verify) sees no dependence at all on the Knudsen number. The simplicity of the asymptotic behavior at large distances from a sphere in free flow is a consequence of the fact that the flow is to a very good approximation that described by the inviscid continuum equations with a point singularity replacing the sphere.

First let us examine the asymptotic behavior macroscopically. To be precise, we consider a scale of length on which not only a, but even L is small. There is a general result for the Boltzmann equation that if we allow the mean-free-path to approach zero in a given problem, the solution to the Boltzmann equation becomes locally Maxwellian and the flow parameters satisfy the conventional non-dissipative equations of gas dynamics locally, i.e. except in certain narrow regions such as shock waves and boundary layers (Ref. (6)). Of course, this could be trivially true in our problem, with the appropriate solution of the fluid equations as the unperturbed constant flow. This would be the case, for example, if the only effect of collisions on the emerging stream were to retard it and produce a flow perturbation which decayed, let us say, exponentially rather than as $1/r^2$. This, however, cannot be. However else the sphere perturbs the flow, it must behave like a momentum sink; the drag is a given constant independent of the distance at which one observes. But this singularity is macroscopic and must make an appearance in the momentum equation. In particular, we discover that the macroscopic solution decays at a rate slower than $1/r^2$ in certain regions.

On the microscopic level it is possible to improve the free flow solution by taking into account successive collisions between the incident and emergent streams, (see Ref. (2), also (3), (4) and additional Refs. quoted there). All such procedures, whether expressed in terms of iterations or as power series in (a/L), seem to give suitable approximations only for distances not too far from the body. This is consistent with the actual picture (which we shall verify) that the effect of the emergent stream is felt at a considerable distance away from the body. Qualitatively, one can visualize a "pile-up" on the order of one mean-free-path in front of the body where the emergent stream begins to deflect the incident stream and is itself stopped and eventually turned back to flow around the body. A number of mean-free-paths beyond the body, this will have developed into a weak shock (strength on the order of a^2/L^2) if the flow is supersonic. From the microscopic picture it is not at all clear why there should be a radical change in the behavior of the flow far from the object when the speed changes from subsonic to supersonic. This distinction can only be made some distance from the body after the continuum equations take over. Similarly, directly behind the body there is a wake which is only partially filled in (residue on the order of a^2/L^2) several mean-free-paths downstream, and its subsequent behavior is macroscopic.

In the remainder of the paper we consider first the elementary problem of the linearized perturbation induced by a point source using the compressible inviscid Euler equations and then show that this solution is the correct limiting flow far from a small object in free flow obtained from the Boltzmann equation. Of course, the Boltzmann solution yields additional information about the flow.

The problem of the flow around an object has been studied in great detail using the Navier-Stokes equations (e.g. Refs. (7), (8), (9), (10)). The first solution we obtain (which is essentially the fundamental solution of the linearized Euler equations) is more primitive and can be obtained from the viscous solutions in the limit as the viscosity approaches zero. The Boltzmann solution is more general and can be used to justify the asymptotic validity of the non-dissipative solution and also to spell out the exact regions of validity of the Navier-Stokes solution.

The solution which we obtain for the flow far from a small body in free flow can be applied to the asymptotic behavior far from a body of arbitrary size. This solution is, to a certain order, independent of the Knudsen number, and it is uniformly valid for all Mach numbers. As a matter of fact, the asymptotic flow is completely determined by a single parameter, the drag on the body.

If ionization and electromagnetic phenomena are taken into account, the picture is much more complicated in detail, but the overall behavior should be quite similar. Suppose we have a body which is small compared to all the relevant lengths, viz. mean-free-paths, Debye length, Larmor radius, and magnetic shielding lengths. The solution will be an elementary free flow near the body. One passes through all sorts of complex transitional zones corresponding to the reference lengths itemized above. Far away one recovers the continuum equations; moreover, at sufficiently large distances these equations will be non-dissipative (including infinite electrical conductivity). This asymptotic solution can be obtained from an elementary magneto-hydrodynamic analysis as well as from an asymptotic Boltzmann equation analysis, but this question will be left to a later publication.

3. THE CONTINUUM SOLUTION

We consider the equations of motion of a compressible gas linearized about a constant density ρ_o; pressure p_o, and velocity u_o:

$$(3.1)\qquad \begin{cases} (u_o \cdot \nabla)\rho + \rho_o \operatorname{div} u = \alpha\rho_o u_o \\ \rho_o(u_o \cdot \nabla)u + \nabla p = -\vec{u}_o(\beta\rho_o u_o) \\ (u_o \cdot \nabla)(p - a_o^2\,\rho) = \gamma\rho_o u_o^3 \end{cases}$$

For generality, arbitrary sources of mass, momentum, and entropy have been included; they will be specified later. The momentum source is taken in the direction of the unperturbed flow. The quantity a_o^2 is the sound speed of the unperturbed flow. The normalization factors, ρ_o and u_o, have been inserted for simplicity of the ensuing formulas; α, β, and γ are dimensionally reciprocal lengths.

An equation for ρ alone is obtained by subtracting the mass and momentum equations after operating with $(u_o \cdot \nabla)$ and div respectively; p is eliminated by use of

$$(3.2)\qquad p = a_o^2\,\rho + \rho_o u_o^2 \gamma^*$$

where

$$(3.3)\qquad \gamma^* = \int_{-\infty}^{x} \gamma\, dx$$

(we take the x-coordinate in the direction u_o). This equation is

$$(3.4)\qquad \frac{1}{\rho_o}\,\Box\rho = -M^2\left(\frac{\partial\alpha}{\partial x} + \frac{\partial\beta}{\partial x} + \Delta\gamma^*\right)$$

where

$$(3.5)\begin{cases} \Delta \equiv \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \\ \Box \equiv (1-M^2)\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2} \\ M^2 = u_o{}^2/a_o{}^2 \end{cases}$$

It is convenient to write

$$(3.6) \qquad \Delta\gamma^* = \Box\gamma^* + M^2\,\partial\gamma/\partial x$$

in (3.4) to obtain

$$(3.7) \qquad \Box(\rho/\rho_o + M^2\gamma^*) = -M^2\frac{\partial}{\partial x}(\alpha + \beta + M^2\gamma)$$

We introduce a solution ϕ of the auxiliary equation

$$(3.8) \qquad \Box\phi = \alpha + \beta + M^2\gamma$$

in terms of which ρ is given by

$$(3.9) \qquad \rho/\rho_o = -M^2(\gamma^* + \frac{\partial\phi}{\partial x})$$

The velocity is then found from

$$\begin{aligned} \rho_o u_o \frac{\partial\vec{u}}{\partial x} &= -\nabla p - \vec{u}_o(\beta\rho_o u_o) \\ &= -\nabla(a_o^2\rho + \rho_o u_o^2\gamma^*) - \vec{u}_o(\beta\rho_o u_o) \\ &= \rho_o u_o^2\frac{\partial\phi}{\partial x} - \vec{u}_o(\beta\rho_o u_o) \end{aligned}$$

or

$$(3.10) \qquad \vec{u} = u_o\nabla\phi - \vec{u}_o\beta^*$$

where

$$(3.11) \qquad \beta^* = \int_\infty^x \beta dx.$$

The relevant singularities are δ-functions for the momentum source and the entropy source

$$(3.12)\quad \begin{cases} \beta = \beta_1 \delta(x,y,z) \\ \gamma = \gamma_1 \delta(x,y,z)\ ; \end{cases}$$

β_1 and γ_1 are dimensionally areas.

For the mass source, the singularity might be a dipole

$$(3.13)\qquad \alpha = \alpha_1 \frac{\partial}{\partial x} \delta(x,y,z),$$

but we shall omit this for the present.

We must now distinguish subsonic flow, $M < 1$, from supersonic flow, $M > 1$. In the former case, $M < 1$, we introduce the fundamental solution of the distorted Laplace equation

$$(3.14)\quad \begin{cases} \Box \psi = \delta \\ \psi = - \dfrac{1}{4\pi r_M} \\ r_M^2 = x^2 + (1-M^2)(y^2+z^2). \end{cases}$$

With the combined momentum and entropy source, we take $\emptyset = (\beta_1 + M^2\gamma_1)\psi$ and the solution is

$$(3.15)\quad \begin{cases} \rho/\rho_o = -M^2\gamma_1\delta^* - M^2(\beta_1 + M^2\gamma_1)\partial\psi/\partial x \\ p = -\rho_o u_o^2(\beta_1 + M^2\gamma_1)\partial\psi/\partial x \\ \vec{u} = -\beta_1\vec{u}_o\delta^* + u_o(\beta_1 + M^2\gamma_1)\nabla\psi \end{cases}$$

The expression

$$(3.16)\qquad \delta^* = \int_{-\infty}^{x} \delta(x,y,z)dx = H(x)\delta(y,z)$$

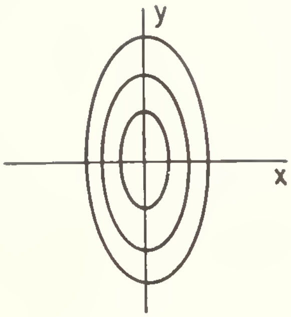

FIG. 1a

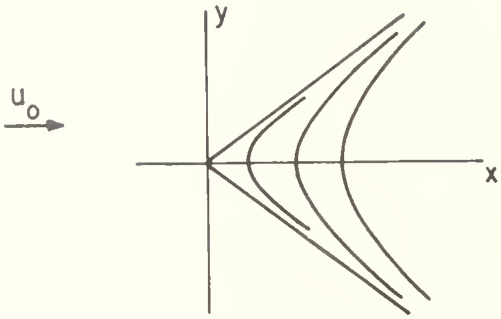

FIG. 1b

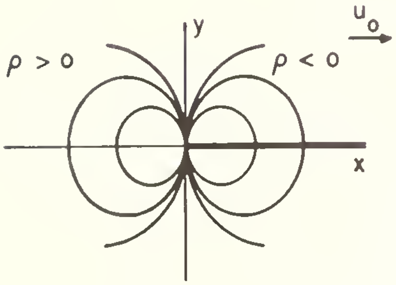

FIG. 2a

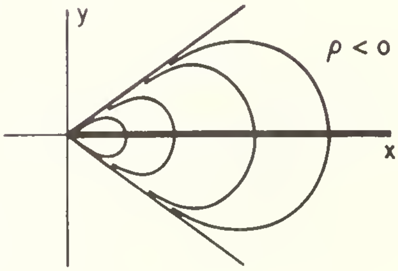

FIG. 2b

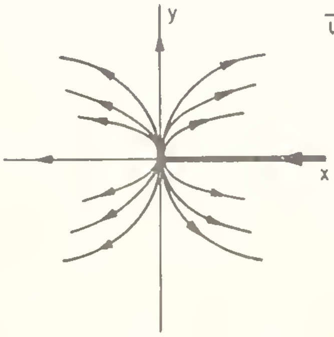

FIG. 3a

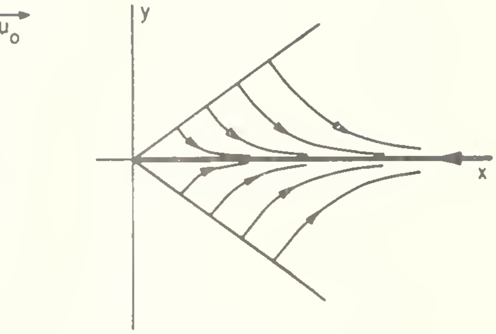

FIG. 3b

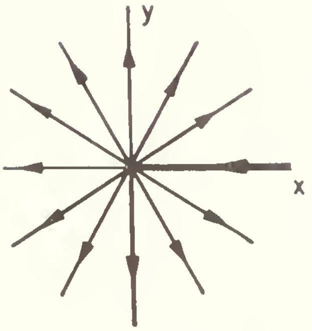

FIG. 4a

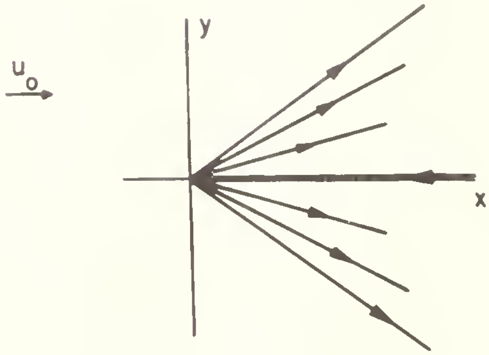

FIG. 4b

represents a singularity on the positive half of the x-axis, $x > 0$. It is a δ-function in y and z for $x > 0$ and it is zero for $x < 0$; ($H(x)$ is the Heaviside function). This singularity represents a wake behind the object. The entropy source produces a wake in the density (also the temperature) and the momentum source produces a wake in the velocity. The pressure is continuous (except, of course, at the origin). The regular part of the solution, expressed in terms of ψ, is analytically the same for the momentum source and for the entropy source. In Fig. 1a we show the contours r_M = constant, in Fig. 2a the contours ρ = constant, and in Fig. 3a the stream lines of u. It is interesting to note that the mass flow vector field, $\rho_o u + \rho u_o$, is the superposition of a radial (isotropic) source and a flow on the positive x-axis, Fig. 4a. This subsonic flow is not symmetric as are many solutions of inviscid problems; the reason is, of course, the momentum source which distinguishes upstream from downstream.

For the supersonic case, $M > 1$, the analysis is quite similar. We use the fundamental solution

$$(3.17) \qquad \begin{cases} \Box \psi = \delta \\ \psi = -\dfrac{1}{2\pi r_M} \\ r_M^2 = x^2 - (M^2-1)(y^2+z^2) \end{cases}$$

To be precise, ψ has the given value in the cone $r_M^2 > 0$, $x > 0$, and it is zero elsewhere. No confusion should arise from the use of the same symbol ψ as before. Taking an entropy and momentum source as before, we have $\phi = (\beta_1 + M^2\gamma_1)\psi$, and we see that the solution is formally identical to the previous one, equation (3.15), noting only the change in the meaning of ψ. There is one point of caution however. This solution must be interpreted as a distribution rather than as a conventional function. It is not only the appearance of the term δ^*, but also the non-integrable expressions $\partial\psi/\partial x$ and $\nabla\psi$ which must be interpreted as distributions. Moreover, the singularities of these functions (which occur on the characteristic cone, $r_M = 0$) are not of the simple type that can be expressed as a δ-function or its derivatives. These complications disappear if the source terms are smeared out to a finite size. One must remember to interpret the solution in terms of distributions when performing the smearing.

The contours r_M = constant and ρ = constant are shown in Figs. 1b and 2b respectively, and the velocity field, u, and mass flow field, $\rho_o u + \rho u_o$, are given in Figs. 3b and 4b respectively. The mass flow field is radial, but it is not isotropic as it was in the subsonic case.

It must be recalled that these solutions are the perturbations over the original ρ_o and u_o. It is somewhat disturbing to obtain a "small" perturbation which is not only infinite at the characteristic cone but even non-integrably infinite, but this difficulty is resolved when we consider the finite size of the source.

4. SOLUTION USING THE BOLTZMANN EQUATION

The physical problem is the flow around an object which is small compared to the mean-free-path. Since the latter is the only length parameter which appears in the Boltzmann equation, we can replace this object by a point source. We take a Boltzmann equation with modified collision term (Ref. (11)) and a source term

$$(4.1) \qquad \xi \cdot \frac{\partial f}{\partial x} = \nu(f_o - f) + \sigma(\xi)\delta(x)$$

The collision frequency, ν, is taken to be a constant. The distribution function $f(\xi, x)$ is normalized as a mass density. We have

$$(4.2) \quad \begin{cases} \bar{\rho} = \int f d\xi \\ \bar{\rho}\,\bar{u} = \int \xi f d\xi \\ \bar{p} = \bar{\rho}R\bar{T} = \int \frac{1}{3}(\xi - \bar{u})^2 f d\xi . \end{cases}$$

We introduce the notation

$$(4.3) \qquad f^{(o)}(\xi) = \frac{\rho_o}{(2\pi RT_o)^{3/2}} \exp[-(\xi - u_o)^2/2RT_o]$$

for the unperturbed distribution at infinity and

$$(4.4) \qquad f_o(\xi, x) = \frac{\bar{\rho}}{(2\pi R\bar{T})^{3/2}} \exp[-(\xi - \bar{u})^2/2RT_o]$$

for the locally Maxwellian distribution with parameters $\bar{\rho}$, $\bar{u}$, $\bar{T}$ obtained from $f(\xi, x)$ at the given point x. The source $\sigma(\xi)$ is assumed to be given and is easily related to the given object. Specifically, the radial flow which arises from the source $\sigma(\xi)$ is identified with the free flow perturbation solution; the cone on which this perturbation differs from zero has shrunk to a ray. In the case of a body which is symmetric about the axis u_o, we have no lift force and we can take the same symmetry for the source $\sigma(\xi)$.

We linearize about the absolute Maxwellian $f^{(o)}$ as follows:

$$(4.5)\quad \begin{cases} f = f^{(o)} + g \\ \overline{\rho} = \rho_o + \rho \\ \overline{u} = u_o + u \\ \overline{p} = p_o + p \\ \overline{T} = T_o + T \end{cases}$$

Introducing the relative velocity

$$(4.6)\qquad c = \xi - u_o$$

we have

$$(4.7)\quad \begin{cases} \rho = \int g d\xi \\ \rho_o u + \rho u_o = \int \xi g d\xi \\ \rho_o u = \int c g d\xi \\ p = \int \frac{1}{3} c^2 g d\xi \\ \rho_o RT = \int (\frac{1}{3} c^2 - RT_o) g d\xi \end{cases}$$

Consistent with the linearization, we expand f_o,

$$(4.8)\qquad f_o = f^{(o)} \left\{ 1 + \frac{\rho}{\rho_o} + \left(\frac{c^2}{2RT_o} - \frac{3}{2}\right)\frac{T}{T_o} + \frac{c \cdot u}{RT_o} \right\},$$

and insert this in (4.1)

$$(4.9)\quad \xi \cdot \frac{\partial g}{\partial x} + \nu g = \nu f^{(o)} \left\{ \frac{\rho}{\rho_o} + \left(\frac{c^2}{2RT_o} - \frac{3}{2}\right)\frac{T}{T_o} + \frac{c \cdot u}{RT_o} \right\} + \sigma\delta.$$

The significance of this linearization requires some elaboration. It was stated previously that the free flow perturbation solution decays as $1/r^2$ from the object. This is true of any of the macroscopic quantities, ρ, u, T, but not of f itself. For f we have a perturbation of fixed magnitude, but one which is confined to a smaller cone as r increases. However, we point out that the particular form (4.1) of the Boltzmann equation is linear in f; the nonlinearity occurs only in the appearance of the macroscopic quantities ρ, u, T in the local Maxwellian f_o. Hence the linearization applies beyond several object diameters.

Next we introduce the (largely dimensionless) quantities

$$(4.10)\quad \left\{\begin{array}{l} \tilde{\rho} = \rho/\rho_o, \quad \tilde{T} = T/T_o, \quad \tilde{p} = p/p_o \\ \tilde{u} = u/\sqrt{RT_o}\ , \quad \tilde{u}_o = u_o\ /\sqrt{RT_o} \\ \tilde{\xi} = \xi/\sqrt{RT_o}\ , \quad \tilde{c} = c/\sqrt{RT_o} = \tilde{\xi} - \tilde{u}_o \\ \tilde{g} = (RT_o)^{3/2}\ g/\rho_o \\ \omega = \dfrac{1}{(2\pi)^{3/2}} \exp(-\tfrac{1}{2}\,\tilde{c}^2) = (RT_o)^{3/2}\ f^{(o)}/\rho_o \\ \tilde{\sigma} = RT_o\ \sigma/\rho_o \\ 1/L = \nu/\sqrt{RT_o}\ . \end{array}\right.$$

The only dimensional quantities remaining are $\tilde{\sigma}$ (a reciprocal length) and the mean-free-path, L. Equation (4.9) takes the form

$$(4.11)\qquad \tilde{\xi}\cdot \frac{\partial \tilde{g}}{\partial x} + \frac{1}{L}\,\tilde{g} = \frac{\omega}{L}\left\{\tilde{\rho} + (\tfrac{1}{2}\,\tilde{c}^2 - \tfrac{3}{2})\tilde{T} + \tilde{c}\cdot\tilde{u}\right\} + \tilde{\sigma}\,\delta$$

Instead of (4.7) we have

$$(4.12)\quad \left\{\begin{array}{l} \tilde{\rho} = \displaystyle\int \tilde{g}\,d\tilde{\xi} \\ \tilde{u} = \displaystyle\int \tilde{c}\tilde{g}\,d\tilde{\xi} \\ \tilde{p} = \displaystyle\int \tfrac{1}{3}\,\tilde{c}^2\tilde{g}\,d\tilde{\xi} \\ \tilde{T} = \displaystyle\int (\tfrac{1}{3}\,\tilde{c}^2 - 1)\tilde{g}\,d\tilde{\xi} \end{array}\right.$$

We solve the integro-differential equation (4.11) by Fourier transform; without confusion we may use the same symbol for the function of x and its transform,

$$(4.13) \qquad \emptyset(x) \rightarrow \int e^{ik\cdot x}\, \emptyset(k)dk.$$

For the Fourier transformed equation, we obtain

$$(4.14) \qquad \tilde{g}(\tilde{\xi},k) = \frac{\tilde{\omega}(\tilde{\xi})}{1+iL\tilde{\xi}\cdot k}\left\{\tilde{\rho} + (\tfrac{1}{2}\,\tilde{c}^2 - \tfrac{3}{2})\tilde{T} + \tilde{c}\cdot\tilde{u}\right\} + \frac{L\tilde{\sigma}\,(\tilde{\xi})}{1+iL\tilde{\xi}\cdot k}$$

The Fourier-transformed distribution, $\tilde{g}$, is explicit as a function of velocity, and as a function of k it is a linear combination of its moments $\tilde{\rho}$, $\tilde{T}$, and $\tilde{u}$ plus a source term. If we multiply this equation by 1, $\tilde{c}$, and $(\frac{1}{3}\,\tilde{c}^2 - 1)$ successively and then integrate over velocity, we obtain a set of five linear inhomogeneous equations to solve for $\tilde{\rho}$, $\tilde{T}$, and $\tilde{u}$. The ultimate solution will then be obtained by inversion of the Fourier transforms.

The five linear equations are

$$(4.15) \qquad \begin{pmatrix} 1-\int \Omega d\tilde{\xi} & -\int \frac{1}{2}(\tilde{c}^2-3)\Omega d\tilde{\xi} & -\int \tilde{c}_s\Omega d\tilde{\xi} \\ -\int \frac{1}{3}(\tilde{c}^2-3)\Omega d\tilde{\xi} & 1-\int \frac{1}{6}(\tilde{c}^2-3)^2\Omega d\tilde{\xi} & -\int \frac{1}{3}\tilde{c}_s(\tilde{c}^2-3)\Omega d\tilde{\xi} \\ -\int \tilde{c}_r\Omega d\tilde{\xi} & -\int \frac{1}{2}\tilde{c}_r(c^2-3)\Omega d\tilde{\xi} & 1-\int \tilde{c}_r\tilde{c}_s\Omega d\tilde{\xi} \end{pmatrix}\begin{pmatrix}\tilde{\rho}\\ \tilde{T}\\ \tilde{u}\end{pmatrix} =$$

$$= L\begin{pmatrix}\tilde{\sigma}_1\\ \tilde{\sigma}_2\\ \tilde{\sigma}_3\end{pmatrix}$$

where

$$(4.16)\quad \begin{cases} \Omega = \dfrac{\omega}{1+iL\tilde{\xi}\cdot k} \\ \tilde{\sigma}_1 = \displaystyle\int \frac{\tilde{\sigma}\, d\tilde{\xi}}{1+iL\tilde{\xi}\cdot k} \\ \tilde{\sigma}_2 = \displaystyle\int \frac{(\frac{1}{3}\,\tilde{c}^2-1)\tilde{\sigma}\, d\tilde{\xi}}{1+iL\tilde{\xi}\cdot k} \\ \tilde{\sigma}_3 = \displaystyle\int \frac{\tilde{c}\;\tilde{\sigma}\, d\tilde{\xi}}{1+iL\tilde{\xi}\cdot k} \end{cases}$$

The matrix in (4.15) has five rows and columns but is written in an abbreviated notation in which the third "vector" row and column represent three rows and columns respectively with a 3x3 block as the lower right entry. To perform the Fourier inversion explicitly, we consider the limit of small L or large x; these are equivalent since all flow quantities are functions of x/L. To lowest order, approximating Ω by ω, the determinant of the system (4.15) is found to be zero. This is consistent with the fact that the inhomogeneous terms $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$ are multiplied by the small quantity L. To the next order we write

$$(4.17)\quad \Omega = \omega(1-iL\tilde{\xi}\cdot k) = \omega(1-iL\tilde{c}\cdot k - iL\tilde{u}_o\cdot k).$$

It is a simple matter to verify that this approximation to Ω yields the correct asymptotic approximation to the various integrals that must be taken over Ω. Inserting this approximation, the linear system for $\tilde{\rho}$, $\tilde{T}$, $\tilde{u}$ becomes

$$(4.18)\quad \begin{pmatrix} i\tilde{u}_o\cdot k & 0 & ik_s \\ 0 & i\tilde{u}_o\cdot k & \frac{2}{3}ik_s \\ ik_r & ik_r & i\tilde{u}_o\cdot k\ \delta_{rs} \end{pmatrix} \begin{pmatrix} \tilde{\rho} \\ \tilde{T} \\ \tilde{u} \end{pmatrix} = \begin{pmatrix} \tilde{\sigma}_1 \\ \tilde{\sigma}_2 \\ \tilde{\sigma}_3 \end{pmatrix}$$

This matrix is easily inverted, and we find

$$(4.19)\qquad D\begin{pmatrix}\tilde{\rho}\\ \tilde{T}\\ \tilde{u}\end{pmatrix} = \begin{pmatrix} \frac{2}{3}k^2-(\tilde{u}_o\cdot k)^2 & -k^2 & (\tilde{u}_o\cdot k)k_s \\ -\frac{2}{3}k^2 & k^2-(\tilde{u}_o\cdot k)^2 & (\tilde{u}_o\cdot k)k_s \\ (\tilde{u}_o\cdot k)k_r & (\tilde{u}_o\cdot k)k_r & [\frac{5}{3}k^2-(\tilde{u}_o\cdot k)^2]\delta_{rs} - \frac{5}{3}k_r k_s \end{pmatrix}\begin{pmatrix}\tilde{\sigma}_1\\ \tilde{\sigma}_2\\ \tilde{\sigma}_3\end{pmatrix}$$

Here D is the determinant of the matrix in (4.18),

$$(4.20)\qquad D = i(\tilde{u}_o\cdot k)[\tfrac{5}{3}k^2 - (\tilde{u}_o\cdot k)^2].$$

We notice that D represents the Fourier transform of the product of the operator $\Box$ with $(u_o\cdot\nabla)$. This operator is easily inverted without recourse to Fourier transformation, so it remains only to invert the right side of equation (4.19). To do this, we first study the integral

$$(4.21)\qquad S(\tilde{\xi},x) = \frac{1}{(2\pi)^3}\int e^{ik\cdot x}\,\frac{\tilde{\sigma}(\tilde{\xi})}{1+iL\tilde{\xi}\cdot k}\,dk$$

This is the Fourier inversion of a typical expression which arises in the source terms $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, and $\tilde{\sigma}_3$, but we have chosen to invert the transform before integrating out the velocity. To evaluate this expression, we choose the fixed vector $\tilde{\xi}$ as a coordinate axis, $\tilde{\xi} = (\tilde{\xi}, 0, 0)$ and write $k = (k_1, k_2, k_3)$, $x = (x_1, x_2, x_3)$. Integration with respect to k_1 yields, for $x_1 > 0$

$$(4.22)\qquad S = \frac{1}{L\tilde{\xi}}\,e^{-x_1/L\tilde{\xi}}\,\tilde{\sigma}(\tilde{\xi})\,\frac{1}{(2\pi)^2}\int e^{i(k_2x_2+k_3x_3)}\,dk_2dk_3$$

and zero for $x_1 < 0$. We recognize the remaining integral as a δ-function with respect to x_2 and x_3 and conclude that

(4.23) $$S = \frac{1}{L\tilde{\xi}} e^{-x_1/L\tilde{\xi}} H(x_1)\delta(x_2,x_3)\tilde{\sigma}(\tilde{\xi})$$

Furthermore, for small L, the first factor approaches a δ-function in x_1. This result has a simple intuitive interpretation of $S(\tilde{\xi},x)$ as a radial flow arising from the source function $\tilde{\sigma}(\tilde{\xi})$. This flow is exactly the free-flow solution for the body which corresponds to the source σ modified by an exponential decay representing collisions with a fixed background gas of mean-free-patn L.

The various integrals $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, and $\tilde{\sigma}_3$ are obtained from S by integration over velocity after multiplication by a suitable polynomial function of $\tilde{\xi}$. The result is an approximate δ-function in x (width on the order of L) with a numerical factor depending on the polynomial. First we remark that, in order to conserve mass, we must have

(4.24) $$\int \tilde{\sigma}(\tilde{\xi})d\tilde{\xi} = 0.$$

From this we conclude that σ_1 vanishes in the limit of small L; i.e. there is no δ-function left in this limit. The Fourier inversions of $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$ are given by

(4.25) $$\begin{aligned} \tilde{\sigma}_2 &= \int (\tfrac{1}{3}\tilde{c}^2-1)\, S(\tilde{\xi},x)d\tilde{\xi} \sim \tau_2\, \tilde{u}_o^2\delta(x) \\ \tilde{\sigma}_3 &= \int \tilde{c}\, S(\tilde{\xi},x)d\tilde{\xi} \sim \tau_3\tilde{u}_o\delta(x) \end{aligned}$$

The vector $\tilde{\sigma}_3$ has the direction $\tilde{u}_o$ in virtue of the assumed symmetry of $\tilde{\sigma}(\tilde{\xi})$. If we assume that the total energy of the gas is conserved (i.e. the object has reached an equilibrium temperature and there is no radiation) then

$$(4.26) \qquad \int \tilde{\xi}^2 \, \tilde{\sigma}(\tilde{\xi}) \;=\; 0.$$

In this case, since $\tilde{c}^2 = \tilde{\xi}^2 - 2\tilde{\xi}\cdot\tilde{u}_o + \tilde{u}_o{}^2$, we can write

$$\tilde{\sigma}_2 = -\frac{2}{3}\int (\tilde{\xi}\cdot\tilde{u}_o) S d\tilde{\xi}$$

but also

$$\tilde{\sigma}_3\cdot\tilde{u}_o = \int (\tilde{\xi}\cdot\tilde{u}_o) S d\tilde{\xi},$$

from which we conclude that

$$(4.27) \qquad \tau_2 = -\frac{2}{3}\tau_3.$$

Returning to (4.19) and adopting the δ-function approximations to $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, and $\tilde{\sigma}_3$ we obtain, after inverting the Fourier transforms,

$$(4.28) \quad \begin{cases} (\tilde{u}_o\cdot\nabla)\Box\tilde{\rho} = [-M^2\tau_2\Box - \frac{3}{5}(M^2\tau_2 - \tau_3)(\tilde{u}_o\cdot\nabla)^2]\delta \\ (\tilde{u}_o\cdot\nabla)\Box\tilde{T} = [M^2\tau_2\Box - \frac{2}{5}(M^2\tau_2 - \tau_3)(\tilde{u}_o\cdot\nabla)^2]\delta \\ (\tilde{u}_o\cdot\nabla)\Box\tilde{u} = [\tau_3\tilde{u}_o\Box + (M^2\tau_2 - \tau_3)(\tilde{u}_o\cdot\nabla)\nabla]\delta \end{cases}$$

We have used the relation

$$(4.29) \qquad \tilde{u}_o = u_o/\sqrt{RT_o} = \sqrt{5/3}\, M$$

Solving these differential equations just as in the previous section, we obtain the identical results that were obtained there (viz. equation (3.15)) provided that we make the identifications

$$(4.30) \quad \begin{cases} \beta_1 = -\tau_3/\tilde{u}_o \\ \gamma_1 = \tau_2/\tilde{u}_o. \end{cases}$$

The restriction (4.27) which arises from energy conservation relates the momentum and entropy sources, (4.31) $\gamma_1 = \frac{2}{3} \beta_1$.

This relation is equivalent to the thermodynamic statement that the entropy source in the wake is given by the irreversible work corresponding to the product of the drag by the free stream velocity, u_o.

It is also easy to demonstrate another universal relation, viz. that the dipole component of the mass source is simply related to the drag, whatever form $\sigma(\xi)$ may take.

5. SUMMARY AND CONCLUSIONS

In the asymptotic approximation for small mean-free-path which we found to yield the continuum result, the finite size of the mean-free-path enters in two distinct places. First there is the finite extent of the source terms σ_1, σ_2, σ_3. Then there is the modification of the equations of motion (i.e. of the matrix of the linear system for ρ, T, u) to include dissipation.

The significant fact about the mass, momentum, and entropy sources is that their extent is measured by the mean-free-path rather than by the object size (which is even zero in this formulation). The sources σ_1, σ_2, σ_3 are finite in extent even though the original source, $\sigma\,\delta$, in the Boltzmann equation is not. Both the shock and the wake are opened up to at least a mean-free-path in thickness. Also, the structure of these sources yields a definite internal structure for the shock and for the wake. This refinement can be easily computed in terms of the fundamental solution ψ (equations (3.14) and (3.17)) of the non-dissipative theory.

This elaboration of the elementary results will yield a mass source which is different from zero; of course the integrated source remains zero, but it is only in the limit $L = 0$ that there is no mass source at all.

Incidentally, if the magnitude of the source, σ, is related to an assumed object of size a which is small compared to L, then the actual disturbance in the shock wave and in the wake will be small (at most of order a^2/L^2) once these boundary layers are smeared out to the thickness L.

The finite extent of the sources yields a minimum thickness for the shock and for the wake. In addition, one naturally expects that keeping a finite mean-free-path in the rest of the theory will produce a progressive widening of these boundary layers. Once the boundary layers thicken to more than a few mean-free-paths, it is natural to expect the usual dissipative continuum theories to apply. As a matter of fact, it is easy to verify that the stresses and heat flow computed for the zero order distribution function found in the last section are exactly the Navier-Stokes values. Of course, the exact nature of the "next" approximation depends on how one decides to improve the Fourier inversion. Nevertheless, one can confidently expect these boundary layers to widen parabolically as would be predicted by the usual viscous and heat conduction laws. This widening is the ultimate justification for the non-dissipative solution as a δ-function in the wake and a sharp singularity on the Mach cone. On a large scale, a parabolically widening boundary layer will appear as a singular line or surface. To be precise, at a fixed position in space, the dissipative solution will approach the non-dissipative one as L -> 0 (the exact solution is a function of x/L). The strength of the singularity δ^* in the wake is a measure of the integrated disturbance over the entire wake. The flow near the regions of nonuniform convergence can be found by allowing (x,y,z) to approach infinity on an appropriate parabolic path instead of radially; this has not yet been done.

It should be remarked that the singularity at the Mach cone should not really be called a shock. By the term "shock" one usually refers to a wave which has reached an equilibrium against the competing effects of dissipation tending to widen it and nonlinearities tending to steepen it. It is easy to verify that the strength of a wave lying on the Mach cone decays at least as fast as 1/r. This would imply a shock thickness increasing at least as r, i.e. conically. But the maximum thickness caused by dissipative broadening is parabolic and has the order $\sqrt{r}$. Thus the broadening process is too slow to ever develop into a true shock. Alternatively, one can say that nonlinearity becomes increasingly negligible with distance from the origin. Also, the terminology "detached" shock is used only because of a general qualitative similarity between the flows. For a true shock, the strength directly ahead of the object is determined by the Mach number alone. In the flows we describe, with the object size, a, small compared to L, the strength of the disturbance ahead of the object cannot greatly exceed the small quantity a^2/L^2. Nevertheless, one can expect a continuous transition of the flow described here into a true continuum flow with a detached shock as a/L increases.

We return to the question, how to improve the elementary results given above. In the neighborhood of the object, the conventional free flow solution is accurate. Far away and outside the boundary layers the elementary continuum flow is valid. One point is abundantly clear. The direct continuum analysis is much simpler than the equivalent asymptotic limit, $L \to 0$ even with the simplified Boltzmann collision term which was adopted. One might wish to improve the continuum model by taking the Navier-Stokes or thirteen moment equations, judiciously modifying the source terms as suggested by the Boltzmann equation analysis. Several steps seem natural. The first is, as mentioned above, to use the same non-dissipative equations but with the smeared out source terms; this answer is explicit as an integral over the source term with the fundamental solution, ψ, as a kernel. Next one might try the Navier-Stokes equations with the same source terms; or the thirteen moment equations, incorporating additional sources in the stress and heat equations by taking second and third moments of the function S (equation (4.23)). Either of these can be expected to give results which are reasonably accurate over the entire range of distances.

There is one other type of improvement which we suggest for its inherent illumination of the problem rather than for its practicability. The same function S which arose in the analysis of the asymptotic behavior as $L \to 0$ arises in another context, viz. in the treatment of the free flow problem near the body, which can be considered as the asymptotic behavior as L $\rightarrow \infty$! The function S is exactly the free flow perturbation (i.e. the difference between the unperturbed flow and the free flow solution--this includes the removed particles in the shadow as well as those reflected by the body) modified by collisional losses with the oncoming stream. This function can be interpreted as the first step in an iteration process; the next step would include as sources the particles whose loss by collision is inherent in the exponential decay of the function S. One could then choose a higher order step in this iteration about the free flow solution as an appropriate source term instead of S itself. In this way, the solution would be simultaneously improved from both ends, near the body and far from it.

This procedure can be generalized. Let us suppose that the distribution function is known exactly (or to some approximation) in any region, small or large, which surrounds a source or a finite body. We replace this region by a fictitious distributed source located on the boundary of the region such that the source exactly duplicates the molecular inflow and outflow. At a large distance from this region, the source can be considered to be a point source in the Boltzmann equation, i.e. it is exactly the function σ. It is clear that the same argument applies to the replacement of a rigid object of any size by an appropriate source. The difficulty is that, if the object is not in free flow, the equivalent source term, σ, is not known but depends on a solution for the distribution function near the object. To the lowest order, however, far from the body, we have seen that a single number, the value of the momentum source or drag, suffices to specify the asymptotic flow. We conclude, therefore, that the elementary solution obtained in Section 3 represents the correct asymptotic flow far from an arbitrary body of any size and speed.

Finally, we conclude with the claim that an entirely equivalent analysis including all of the complexities associated with an ionized conducting gas seems to be quite feasible. In particular, one can expect that the simplest non-dissipative continuum equations will give the correct asymptotic behavior. Unfortunately, this continuum theory can only be solved explicitly in certain special cases, but approximation techniques to cover the entire problem are being developed and should be at hand shortly.

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